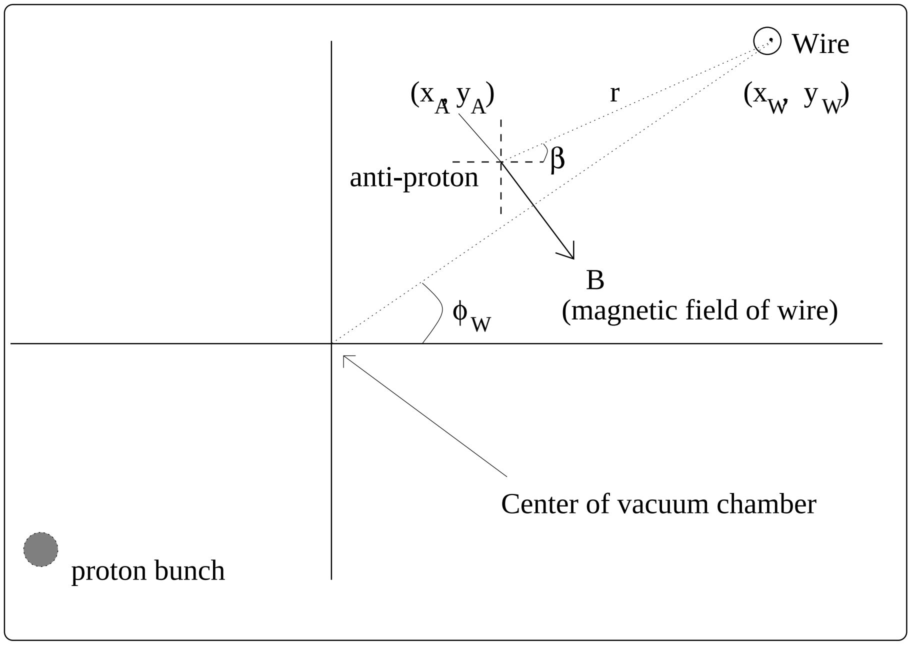

Figure 1: Coordinates of the wire, test anti-proton and proton bunch. Current in the wire is directed out of the plane of the paper.

1 Ideal Compensation (1)

We consider the opposing proton beam to be round in this section. If the coordinates of the proton centroid are (x_{P0}, y_{P0}) and the coordinates of the test anti-proton particle are (x_A, y_A), the beam-beam kicks experienced by the anti-proton are

$$\Delta x' = -\frac{2N_p r_p}{\gamma_p} \frac{x_{P0} - x_A}{[(x_{P0} - x_A)^2 + (y_{P0} - y_A)^2]} \left\{ 1 - \exp[-\frac{1}{2\sigma^2}[(x_{P0} - x_A)^2 + (y_{P0} - y_A)^2]] \right\} \quad (1)$$

$$\Delta y' = -\frac{2N_p r_p}{\gamma_p} \frac{y_{P0} - y_A}{[(x_{P0} - x_A)^2 + (y_{P0} - y_A)^2]} \left\{ 1 - \exp[-\frac{1}{2\sigma^2}[(x_{P0} - x_A)^2 + (y_{P0} - y_A)^2]] \right\} \quad (2)$$

`CHECK SIGNS`

If the two beams are sufficiently far away that the argument of the exponential is $\gg 1$, then the exponential term has a negligible contribution and only the first term in { } need be kept.

Now consider the field and forces due to a wire on the test anti-proton. For an infinitely long wire, the magnetic field is azimuthally directed and at a distance r from the wire is given by

$$B_\theta = \frac{\mu_0}{2\pi} \frac{I_W}{r} \quad (3)$$

where I_W is the current in the wire.

From Figure 1 it follows that the components of the field are

$$B_x = |B| \sin\beta = \frac{\mu_0}{2\pi} \frac{y_W - y_A}{r^2} I_W \ , \qquad B_y = -|B| \cos\beta = \frac{\mu_0}{2\pi} \frac{x_W - x_A}{r^2} I_W \quad (4)$$

Here

$$r^2 = (x_W - x_A)^2 + (y_W - y_A)^2$$

Assuming that the force due to the wire can be lumped in the middle of the wire and considered as that due to an impulsive force, the change in slopes of the anti-proton are

$$\Delta x' = -\frac{B_y L}{(B\rho)} = \frac{\mu_0}{2\pi}\frac{I_W}{(B\rho)}\frac{x_W - x_A}{(x_W - x_A)^2 + (y_W - y_A)^2} \tag{5}$$

$$\Delta y' = \frac{B_x L}{(B\rho)} = \frac{\mu_0}{2\pi}\frac{I_W}{(B\rho)}\frac{y_W - y_A}{(x_W - x_A)^2 + (y_W - y_A)^2} \tag{6}$$

Convert to Floquet coordinates

$$\begin{aligned} X &= \frac{x}{\beta_x}, \quad P_X = \frac{\beta_x x' + \alpha_x x}{\sqrt{\beta_x}} \\ Y &= \frac{y}{\beta_y}, \quad P_Y = \frac{\beta_y y' + \alpha_y y}{\sqrt{\beta_y}} \end{aligned} \tag{7}$$

If the motion between points 1 and 2 is purely linear, then the transformation between the Floquet coordinates between these two points is

$$\begin{pmatrix} X \\ P_X \\ Y \\ P_Y \end{pmatrix}_2 = \begin{bmatrix} R(\psi_x) & 0 \\ 0 & R(\psi_y) \end{bmatrix} \begin{pmatrix} X \\ P_X \\ Y \\ P_Y \end{pmatrix}_1 \tag{8}$$

where $R(\psi)$ is the 2×2 rotation matrix.

The impulsive kicks in Floquet coordinates are given by

$$\Delta P_X = \sqrt{\beta_x}\Delta x' , \qquad \Delta P_Y = \sqrt{\beta_y}\Delta y'$$

The beta functions that will be used are those on the *pbar* helix, even when scaling the coordinates of the *proton bunch*. Thus

$$\begin{aligned} X_{P0} &= \frac{x_{P0}}{\sqrt{\beta_x(\bar{p})}}, \quad X_A = \frac{x_A}{\sqrt{\beta_x(\bar{p})}} \\ Y_{P0} &= \frac{y_{P0}}{\sqrt{\beta_y(\bar{p})}}, \quad Y_A = \frac{y_A}{\sqrt{\beta_y(\bar{p})}} \end{aligned} \tag{9}$$

Defining the constants

$$C_b = \frac{2N_p r_p}{\gamma_p}, \quad C_W = \frac{\mu_0}{2\pi}\frac{I_W L}{(B\rho)} \tag{10}$$

the kicks due to the round beam-beam interaction and the wire can be written as

$$\begin{aligned} \Delta P_{X,b} &\simeq -C_b \frac{\beta_{x,b}(X_{P0,b} - X_{A,b})}{[\beta_{x,b}(X_{P0,b} - X_{A,b})^2 + \beta_{y,b}(Y_{P0,b} - Y_{A,b})^2]} \\ \Delta P_{Y,b} &\simeq -C_b \frac{\beta_{y,b}(Y_{P0,b} - Y_{A,b})}{[\beta_{x,b}(X_{P0,b} - X_{A,b})^2 + \beta_{y,b}(Y_{P0,b} - Y_{A,b})^2]} \end{aligned} \tag{11}$$

$$\begin{aligned} \Delta P_{X,W} &= C_W \frac{\beta_{x,W}(X_W - X_{A,W})}{\beta_{x,W}(X_W - X_{A,W})^2 + \beta_{y,W}(Y_W - Y_{A,W})^2} \\ \Delta P_{Y,W} &= C_W \frac{\beta_{y,W}(Y_W - Y_{A,W})}{\beta_{x,W}(X_W - X_{A,W})^2 + \beta_{y,W}(Y_W - Y_{A,W})^2} \end{aligned} \tag{12}$$

where $(\beta_{x,b}, \beta_{y,b})$, $(X_{P0,b}, Y_{P0,b}, X_{A,b}, Y_{A,b})$ are the beta functions on the pbar helix and coordinates at the beam-beam interaction while $(\beta_{x,W}, \beta_{y,W})$, $(X_W, Y_W, X_{A,W}, Y_{A,W})$ are the beta functions also on the pbar helix and coordinates of the wire and test pbar particle at the wire location.

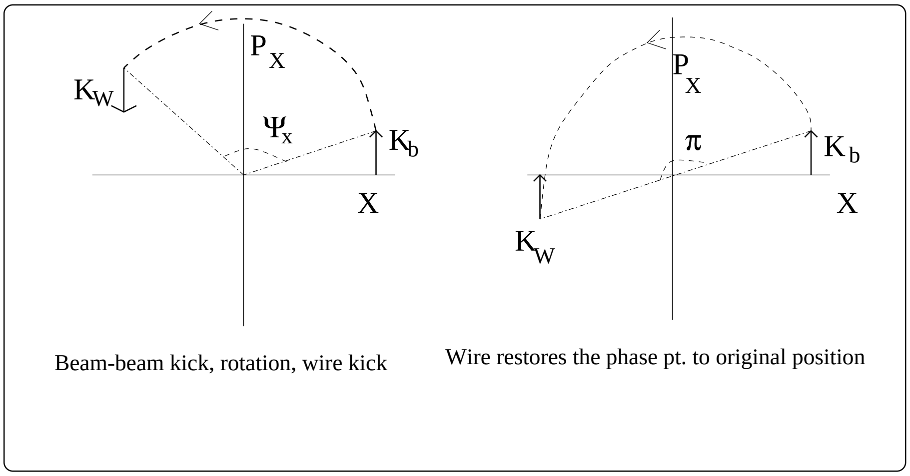

Figure 2: Phase space point following a beam-beam kick, phase rotation and followed by a wire.

1.1 One beam-beam kick

Let $\vec{Z} = [X, P_X, Y, P_Y]^T$ denote the phase space vector in Floquet space. In this model where there is only linear motion between the beam-beam kick and the wire, the phase space vector is

$$\vec{Z}_f = K_W \odot R(\psi_x, \psi_y) \odot K_b \odot \vec{Z}_i \tag{13}$$

Principle of Compensation: The wire should restore the phase space trajectory to the point reached in the absence of the beam-beam interaction and the wire. In other words, the phase point after the wire is the same as though the motion were completely linear.

This requires

$$K_W \odot R(\psi_x, \psi_y) \odot K_b \odot \vec{Z}_i = R(\psi_x, \psi_y) \odot \vec{Z}_i \tag{14}$$

or equivalently

$$\begin{aligned} \sin\psi_x \Delta P_{X,b} &= 0 && (15) \\ \cos\psi_x \Delta P_{X,b} + \Delta P_{X,W} &= 0 && (16) \\ \sin\psi_y \Delta P_{Y,b} &= 0 && (17) \\ \cos\psi_y \Delta P_{Y,b} + \Delta P_{Y,W} &= 0 && (18) \end{aligned}$$

Equations (15) and (17) determine the phase at which the wire should be placed. It is obvious that since the wire changes only the momenta but not the instantaneous positions, the phases should be chosen so that the positions have the values that they would in the absence of the kicks. Indeed we find

$$\psi_x = m_x \pi, \qquad \psi_y = m_y \pi \tag{19}$$

The other two equations (16) and (18) determine the wire parameters (current and positions) so that the momenta are also returned to their original values.*In general, this compensation only works for a specified test anti-proton.* Suppose we choose this compensation to be effective on the centroid of the anti-protons with coordinates $(X_{A0}, P_{X,A0}, Y_{A0}, P_{Y,A0})$.

We have

$$\frac{\beta_{x,W}(X_W - X_{A,W})}{\beta_{x,W}(X_W - X_{A,W})^2 + \beta_{y,W}(Y_W - Y_{A,W})^2} = -\cos\psi_x \frac{\Delta P_{X,b}(A0)}{C_W} \equiv C_{X,A0} \tag{20}$$

$$\frac{\beta_{y,W}(Y_W - Y_{A,W})}{\beta_{x,W}(X_W - X_{A,W})^2 + \beta_{y,W}(Y_W - Y_{A,W})^2} = -\cos\psi_y \frac{\Delta P_{Y,b}(A0)}{C_W} \equiv C_{Y,A0} \tag{21}$$

These have the solutions for the wire positions (X_W, Y_W)

$$X_W - X_{A0,W} = \frac{\beta_{y,W} C_{X,A0}}{\beta_{y,W} C_{X,A0}^2 + \beta_{x,W} C_{Y,A0}^2}, \qquad Y_W - Y_{A0,W} = \frac{\beta_{x,W} C_{Y,A0}}{\beta_{y,W} C_{X,A0}^2 + \beta_{x,W} C_{Y,A0}^2} \tag{22}$$

This fixes the wire position in terms of the beam-beam kicks experienced by the bunch centroid. It places no restriction as yet on the wire current or other optics constraints at the location of the wire.

The condition that the compensation works for any other particle with Floquet coordinates (X_A, Y_A) in the anti-proton bunch is

$$X_W - X_{A,W} = \frac{\beta_{y,W} C_{X,A}}{\beta_{y,W} C_{X,A}^2 + \beta_{x,W} C_{Y,A}^2}, \quad Y_W - Y_{A,W} = \frac{\beta_{x,W} C_{Y,A}}{\beta_{y,W} C_{X,A}^2 + \beta_{x,W} C_{Y,A}^2} \tag{23}$$

where the position of the wire (X_W, Y_W) is determined by Equation (22). We can write for example

$$\begin{aligned} X_W - X_{A,W} &= X_W - X_{A0,W} - (X_{A,W} - X_{A0,W}) \\ &= \frac{\beta_{y,W} C_{X,A}}{\beta_{y,W} C_{X,A}^2 + \beta_{x,W} C_{Y,A}^2} + + \left\{ \frac{\beta_{y,W} C_{X,A0}}{\beta_{y,W} C_{X,A0}^2 + \beta_{x,W} C_{Y,A0}^2} - \frac{\beta_{y,W} C_{X,A}}{\beta_{y,W} C_{X,A}^2 + \beta_{x,W} C_{Y,A}^2} - (X_{A,W} - X_{A0,W}) \right\} \end{aligned}$$

The terms in { } must vanish for exact compensation or

$$\frac{\beta_{y,W} C_{X,A0}}{\beta_{y,W} C_{X,A0}^2 + \beta_{x,W} C_{Y,A0}^2} - \frac{\beta_{y,W} C_{X,A}}{\beta_{y,W} C_{X,A}^2 + \beta_{x,W} C_{Y,A}^2} = X_{A,W} - X_{A0,W} = \cos(\psi_x)[X_{A,b} - X_{A0,b}] \tag{24}$$

Writing

$$\Delta X_A = X_{P0,b} - X_{A,b}, \quad \Delta Y_A = Y_{P0,b} - Y_{A,b}, \quad \Delta R_A^2 = \beta_{x,b} \Delta X_A^2 + \beta_{y,b} \Delta Y_A^2, \quad C_r = \frac{C_b}{C_W}$$

we have

$$\begin{aligned} \frac{\beta_{y,W} C_{X,A0}}{\beta_{y,W} C_{X,A0}^2 + \beta_{x,W} C_{Y,A0}^2} - \frac{\beta_{y,W} C_{X,A}}{\beta_{y,W} C_{X,A}^2 + \beta_{x,W} C_{Y,A}^2} &= \frac{\cos\psi_x \Delta X_{A0} \Delta R_{A0}^2}{C_r[\beta_{x,b} \Delta X_{A0}^2 + (\beta_{y,b}^2/\beta_{x,b})(\beta_{x,W}/\beta_{y,W})\Delta Y_{A0}^2]} \\ &\quad - \frac{\cos\psi_x \Delta X_A \Delta R_A^2}{C_r[\beta_{x,b} \Delta X_A^2 + (\beta_{y,b}^2/\beta_{x,b})(\beta_{x,W}/\beta_{y,W})\Delta Y_A^2]} \end{aligned}$$

where we have used the fact that $\cos^2\psi_x = 1 = \cos^2\psi_y$. If $\beta_{y,W}/\beta_{x,W} = \beta_{y,b}/\beta_{x,b}$, then

$$\frac{\beta_{y,W} C_{X,A0}}{\beta_{y,W} C_{X,A0}^2 + \beta_{x,W} C_{Y,A0}^2} - \frac{\beta_{y,W} C_{X,A}}{\beta_{y,W} C_{X,A}^2 + \beta_{x,W} C_{Y,A}^2} = \frac{1}{C_r \cos\psi_x}[X_{A,b} - X_{A0,b}] \tag{25}$$

and the equality to be satisfied is

$$\frac{1}{C_r \cos\psi_x}[X_{A,b} - X_{A0,b}] = X_{A,W} - X_{A0,W} = \cos(\psi_x)[X_{A,b} - X_{A0,b}] \tag{26}$$

This is true if $C_r \equiv C_b/C_W = 1$. The same condition is obtained by requiring that the compensation is exact in the vertical plane for all particles. Hence *the two conditions for the compensation to be exact for all particles are*

$$\frac{\beta_{y,W}}{\beta_{x,W}} = \frac{\beta_{y,b}}{\beta_{x,b}} \tag{27}$$

$$\frac{2N_p r_p}{\gamma_p} = \frac{\mu_0}{2\pi}\frac{I_W L}{(B\rho)} \tag{28}$$

The first condition requires that in addition to the phase advance to the wire being a multiple of π, the beta functions have to be in the same ratio as at the beam-beam interaction. The second condition determines the integrated wire strength in terms of the bunch current.

1.2 Two beam-beam interactions compensated by a single wire

Consider for simplicity that two beam-beam interactions will be compensated by a single wire. The phase advances between the successive beam-beam kicks are $(\psi_{x,1}, \psi_{y,1})$ and the phase advances from the 2nd beam-beam kick to the wire are $(\psi_{x,2}, \psi_{y,2})$.

Applying the same principle of compensation that the wire should restore the phase space point back to the circle after the wire, the compensation condition is

$$K_W \odot R(\psi_{x,2}, \psi_{y,2}) \odot K_{b2} \odot R(\psi_{x,1}, \psi_{y,1}) \odot K_{b1}\vec{Z}_i = R(\sum \psi_x, \sum \psi_y) \odot \vec{Z}_i \tag{29}$$

or equivalently

$$\begin{aligned} \sin \sum \psi_x \Delta P_{X,b1} + \sin \psi_{x,2} \Delta P_{X,b2} &= 0 && (30)\\ \cos \sum \psi_x \Delta P_{X,b1} + \cos \psi_{x,2} \Delta P_{X,b2} + \Delta P_{X,W} &= 0 && (31)\\ \sin \sum \psi_y \Delta P_{Y,b1} + \sin \psi_{y,2} \Delta P_{Y,b2} &= 0 && (32)\\ \cos \sum \psi_y \Delta P_{Y,b1} + \cos \psi_{y,2} \Delta P_{Y,b2} + \Delta P_{Y,W} &= 0 && (33) \end{aligned}$$

Solving for the phases $(\psi_{x,2}, \psi_{y,2})$ to the wire, we find

$$\begin{aligned} \tan \psi_{x,2} &= -\frac{\sin \psi_{x,1}}{[\cos \psi_{x,1} + \Delta P_{x,b2}/\Delta P_{x,b1}]} && (34)\\ \tan \psi_{y,2} &= -\frac{\sin \psi_{y,1}}{[\cos \psi_{y,1} + \Delta P_{y,b2}/\Delta P_{y,b1}]} && (35) \end{aligned}$$

It is no longer necessary that the phase advance from the first beam-beam kick to the wire be a multiple of π.

If we define

$$\begin{aligned} C_x &\equiv \frac{C_{b1}}{C_W}\frac{\cos \sum \psi_x \beta_{x,b1}\Delta X_{A,b1}}{[\beta_{x,b1}\Delta X_{A,b1}^2 + \beta_{y,b1}\Delta Y_{A,b1}^2]} + \frac{C_{b2}}{C_W}\frac{\cos \psi_{x,2}\beta_{x,b2}\Delta X_{A,b2}}{[\beta_{x,b2}\Delta X_{A,b2}^2 + \beta_{y,b2}\Delta Y_{A,b2}^2]}\\ C_y &\equiv \frac{C_{b1}}{C_W}\frac{\cos \sum \psi_y \beta_{y,b1}\Delta X_{A,b1}}{[\beta_{x,b1}\Delta X_{A,b1}^2 + \beta_{y,b1}\Delta Y_{A,b1}^2]} + \frac{C_{b2}}{C_W}\frac{\cos \psi_{y,2}\beta_{y,b2}\Delta Y_{A,b2}}{[\beta_{x,b2}\Delta X_{A,b2}^2 + \beta_{y,b2}\Delta Y_{A,b2}^2]} \end{aligned}$$

then the position of the wire required to compensate the kicks on the chosen anti-proton is given by a similar expression as for a single beam-beam kick

$$X_W - X_{A,W} = \frac{\beta_{y,W} C_{X,A}}{\beta_{y,W} C_{X,A}^2 + \beta_{x,W} C_{Y,A}^2}, \qquad Y_W - Y_{A,W} = \frac{\beta_{x,W} C_{Y,A}}{\beta_{y,W} C_{X,A}^2 + \beta_{x,W} C_{Y,A}^2} \tag{36}$$

This only guarantees that the beam-beam kicks will be compensated for the chosen anti-proton. The conditions for the compensation to work for all anti-protons are much more complicated than with the single beam-beam interaction.

`Comments: From the messy algebra, it doesn't look likely that the beam-beam kicks can be compensated for all particles. Prove or disprove this.`

1.3 Several beam-beam interactions compensated by a single wire

The generalization to N kicks followed by a single wire is straightforward. The phase advances to the wire $(\psi_{x,N}, \psi_{y,N})$ from the N'th beam-beam kick are given by

$$\begin{aligned} \tan \psi_{x,N} &= -\frac{\sin[\sum_{i=1}^{N-1} \psi_{x,i}]\Delta P_{X,1} + \sin[\sum_{i=2}^{N-1} \psi_{x,i}]\Delta P_{X,2} + \ldots + \sin[\psi_{X,N-1}]\Delta P_{X,N-1}}{\left[\cos[\sum_{i=1}^{N-1} \psi_{x,i}]\Delta P_{X,1} + \cos[\sum_{i=2}^{N-1} \psi_{x,i}]\Delta P_{X,2} + \ldots + \cos[\psi_{x,N-1}]\Delta P_{X,N-1} + \Delta P_{X,N}\right]} && (37)\\ \tan \psi_{y,N} &= -\frac{\sin[\sum_{i=1}^{N-1} \psi_{y,i}]\Delta P_{Y,1} + \sin[\sum_{i=2}^{N-1} \psi_{y,i}]\Delta P_{Y,2} + \ldots + \sin[\psi_{Y,N-1}]\Delta P_{Y,N-1}}{\left[\cos[\sum_{i=1}^{N-1} \psi_{y,i}]\Delta P_{Y,1} + \cos[\sum_{i=2}^{N-1} \psi_{y,i}]\Delta P_{Y,2} + \ldots + \cos[\psi_{y,N-1}]\Delta P_{Y,N-1} + \Delta P_{Y,N}\right]} && (38) \end{aligned}$$

The coefficients C_x, C_y are defined as

$$
\begin{aligned}
C_x &= \frac{C_{b1}}{C_W}\frac{\cos[\sum_{i=1}^{N}\psi_{x,i}]\beta_{x,b1}\Delta X_{A,b1}}{[\beta_{x,b1}\Delta X_{A,b1}^2+\beta_{y,b1}\Delta Y_{A,b1}^2]}+\frac{C_{b2}}{C_W}\frac{\cos[\sum_{i=2}^{N}\psi_{x,i}]\beta_{x,b2}\Delta X_{A,b2}}{[\beta_{x,b2}\Delta X_{A,b2}^2+\beta_{y,b2}\Delta Y_{A,b2}^2]}+\dots \\
&\quad +\frac{C_{bN}}{C_W}\frac{\cos[\psi_{x,N}]\beta_{x,bN}\Delta X_{A,bN}}{[\beta_{x,bN}\Delta X_{A,bN}^2+\beta_{y,bN}\Delta Y_{A,bN}^2]} \qquad (39) \\
C_y &= \frac{C_{b1}}{C_W}\frac{\cos[\sum_{i=1}^{N}\psi_{y,i}]\beta_{y,b1}\Delta Y_{A,b1}}{[\beta_{x,b1}\Delta X_{A,b1}^2+\beta_{y,b1}\Delta Y_{A,b1}^2]}+\frac{C_{b2}}{C_W}\frac{\cos[\sum_{i=2}^{N}\psi_{y,i}]\beta_{y,b2}\Delta Y_{A,b2}}{[\beta_{x,b2}\Delta X_{A,b2}^2+\beta_{y,b2}\Delta Y_{A,b2}^2]}+\dots \\
&\quad +\frac{C_{bN}}{C_W}\frac{\cos[\psi_{y,N}]\beta_{y,bN}\Delta Y_{A,bN}}{[\beta_{x,bN}\Delta X_{A,bN}^2+\beta_{y,bN}\Delta Y_{A,bN}^2]} \qquad (40)
\end{aligned}
$$

The solutions for the wire position are then those given in Equation (36).